647510 **MSC4MatC402x** **Seat No :________**

M.Sc.(Maths) Semester - 4 (*CBCS*) Examination
April/May-2022 (NEW COURSE)
Integration Theory (CORE)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1 (A) Answer any Two out of Three. **10**

(1) Let f be an integrable function on the measure space (X, B, μ). Show that given $\epsilon > 0$, there is a $\delta > 0$ such that $\left|\int_E f\, d\mu\right| < \epsilon$ whenever E is a measurable subset of X with $\mu(E) < \delta$.

(2) Let (X, B, μ) be a measure space, and let (f_n) be a sequence of nonnegative measurable functions that converge almost everywhere to a function f. Show that $\int_X f\, d\mu \le \text{liminf}_n \int_X f_n\, d\mu$.

(3) Let (X, B, μ) be a measure space, and let (E_n) be a decreasing sequence of measurable sets with $\mu(E_1) < \infty$. Show that $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$. Show that the conclusion may not hold if $\mu(E_n) = \infty$ for all n

Q.1 (B) Answer any One out of Two. **4**

(1) Let X be an uncountable set, and let η be the counting measure defined on the power set $P(X)$ of X. Show that $(X, P(X), \eta)$ is not a σ- finite measure space.

(2) Let (X, B, μ) be a measure space. If (X, B, μ) is not a complete measure space, then show that there is a sequence (f_n) of measurable functions which converges to some f a.e. on X but f is not measurable.

Q.2 (A) Answer any Two out of Three. **10**

(1) Let (X, B, ν) be a signed measure space, and let E be a measurable subset of X such that $0 < \nu(E) < \infty$. Show that there is a positive set A contained in E with $\nu(A) > 0$.

(2) Let ν be a signed measure on the measurable space (X, B). Show that there exist a unique pair ν^+ and ν^- of two mutually singular measures on (X, B) such that $\nu = \nu^+ - \nu^-$.

(3) If ν_1 and ν_2 are any two finite signed measures on a measurable space (X, B) and $\alpha, \beta \in \mathbb{R}$, then show that $\alpha\nu_1 + \beta\nu_2$ is a signed measure. Also, show that $|\alpha\nu_1| = |\alpha||\nu_1|$.

Q.2 (B) Answer any One out of Two. **4**

(1) Let (X, B, ν) be a signed measure space, and let E be a measurable subset of X such that $\mu(E) > 0$. Show that E need not be a positive set.

(2) Let (X, B, ν) be a signed measure space. If E is any measurable subset of X, then show that $-\nu^-(E) \le \nu(E) \le \nu^+(E)$ and $|\nu(E)| \le |\nu|(E)$.

Q.3 (A) Answer any Two out of Three. **10**

(1) Let (X, B, μ) be a σ-finite measure space and ν be a σ-finite measure defined on B. Show that we can find a measure ν_0, singular with respect to μ, and a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$.

(2) Let μ^* be a outer measure on X. Show that the class B of all μ^*- measurable sets is a σ-algebra.

(3) Let μ be a σ-finite measure on an algebra A, and let μ^* be the outer measure generated by A. If E is a measurable subset of X, then show that $E = P - Q$ for some $A_{\sigma\delta}$ set P and for some set Q satisfying $\mu^*(Q) = 0$.

Q.3 (B) Answer any One out of Two. **4**

(1) Let ν and μ be σ- finite measures on a measurable space (X, B), and let $\nu \ll \mu$. If f is a nonnegative measurable function on X, then prove that $\int_X f\, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu$, where $\left[\frac{d\nu}{d\mu}\right]$ is the Radon-Nikodym derivative of ν with respect to μ.

(2) Let μ be a measure on an algebra A, μ^* the outer measure induced by μ, and E any set. If $\epsilon > 0$, then prove that there is a set $F \in A_\sigma$ with $E \subset F$ and $\mu^* F \leq \mu^* E + \epsilon$.

Q.4 (A) Answer any Two out of Three. **10**

(1) If μ is a finite Baire measure on the real line, then show that its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right. Also show that $\lim_{x \to -\infty} F(x) = 0$.

(2) Let (X, B, μ) be a finite measure space, $1 < p < \infty$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. If F is a bounded linear functional on $L^p(\mu)$, then show that there is an element $g \in L^q(\mu)$ such that $F(f) = \int_X fg\, d\mu$ for all $f \in L^p(\mu)$ and $\|F\| = \|g\|_q$.

(3) Let (X, A, μ) and (Y, B, ν) be σ- finite measure spaces, and let R be the set of all measurable rectangles. Let E be a set in $R_{\sigma\delta}$ with $(\mu \times \nu)(E) < \infty$. Define a function g by $g(x) = \nu(E_x)$ for all $x \in X$. Show that g is measurable and $\int_X g\, d\mu = (\mu \times \nu)(E)$.

Q.4 (B) Answer any One out of Two. **4**

(1) Let $1 < p < \infty$, q be the conjugate index of p, let (X, B, μ) be a finite measure space and g an integrable function such $\left|\int_X g\varphi\, d\mu\right| \leq M\|\varphi\|_p$ for some constant M and for all simple functions φ. Show that $g \in L^q(\mu)$.

(2) Let (X, A, μ) and (Y, B, ν) be σ- finite measure spaces If E is a set for which $(\mu \times \nu)(E) = 0$, then show that $\nu(E_x) = 0$ for almost all x.

Q.5 (A) Answer any Two out of Three. **10**

(1) Let μ be a measure defined on a σ-algebra M containing the Baire sets. Assume either that μ is quasi regular or that μ is inner regular. If $E \in M$ and $\mu(E) < \infty$, then show that there exists a Baire set B such that $\mu(E \Delta B) = 0$.

(2) If μ^* is a topologically regular outer measure on a locally compact Hausdorff space X, then show that each Borel set is μ^*-measurable.

(3) Let λ be a content on a class $\mathcal{K}$ of compact sets of a locally compact Hausdorf space X. Show that there is a unique quasi regular Borel measure $\bar{\mu}$ such that $\breve{\mu}(O) = \sup\{\lambda(K): K \subset O, K \in \mathcal{K}\}$ for every open subset O of X.

Q.5 (B) Answer any One out of Two. **4**

(1) Let μ be a finite measure defined on a σ-algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, then show that it is regular.

(2) State Riesz – Markov Theorem and define each term appearing in the theorem.

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